Pergamon

Int. J. Solids Structures Vol. 31, No. 5, pp. 705–710, 1994

0020–7683/94 $6.00 + .00

NON-LINEAR ANALYSIS OF HEATED RHOMBIC PLATES

AJIT KUMAR RAY
Senior Lecturer in Physics, Alipurduar College, Alipurduar Court, Jalpaiguri, 736122, West Bengal, India

BARUN BANERJEE†
A.D.P.I., Government of West Bengal, Minakshi Bhawan, New Circular Road, Jalpaiguri, 736105, West Bengal, India

and

BISWANATH BHATTACHARJEE
Faculty of Science, North Bengal University, Darjeeling 734430, West Bengal, India

(*Received* 24 *September* 1992; *in revised form* 15 *June* 1993)

Abstract—This paper concerns a new approach to the investigation of non-linear behaviours of heated rhombic plates. A new set of differential equations in oblique co-ordinates have been derived in this investigation. Numerical results showing central deflection parameters versus thermal load functions have been computed for different skew angles θ. For $\theta = 0°$ the results obtained in the present study are in excellent agreement with the known results. It is believed that the results obtained for other different skew angles are completely new.

INTRODUCTION

Determination of thermal deflections in thin elastic plates, is of vital importance in cases where the thermal stresses play a significant role. Although thermal deflections of thin elastic plates have been investigated by many authors (Aleck, 1949; Zizicas, 1952; Schneider, 1955; Boley and Weiner, 1960; Forray and Newmann, 1960; Nowacki, 1962; Katayama *et al.*, 1967; Sarkar, 1968; Kaiuk and Pavlenko, 1971, 1972; Roychowdhury, 1972; Prabhu and Durvasula, 1974; Matumoto and Sekiya, 1975), the literature on the large thermal deflections is somewhat sparse. The most interesting papers in this field are by Williams (1955, 1958) who quite elegantly carried out large deflection analysis for a plate strip subjected to normal pressure and heating. Biswas investigated the large deflection of heated circular plates under non-constant temperature (Biswas, 1974) and large deflections of heated elastic plates under uniform load (Biswas, 1975). The author followed Berger's equation in his investigations. Another interesting paper in this field is by Banerjee and Dutta (1979), in which investigation of non-linear behaviours of heated elastic plates under non-constant temperatures has been carried out. The authors utilized a conformal mapping technique along with Berger's hypothesis. Later on Banerjee proposed a new approach to the Large Deflection analysis of thin elastic plates (Banerjee and Dutt, 1981) and afterwards carried out quite elegantly the non-linear behaviours of polygonal plates under non-constant temperatures (Banerjee, 1984). Following Banerjee's approach, another interesting paper is by Sinharay and Banerjee (1985) on non-linear behaviours of heated spherical and cylindrical shells, where the authors have achieved satisfactory results from the practical point of view. Also, the works of Kamiya (1978) on the large thermal bending of sandwich plates are very attractive and useful too.

All the investigations mentioned above deal with plate geometry either circular or rectangular or in the shape of regular polygons. Only five papers (Katayama *et al.*, 1967; Kaiuk and Pavlenko, 1971, 1972; Prabhu and Durvasula, 1974; Matumoto and Sekiya, 1975) concerned with the study of thermal behaviours of skew plates are found in the literature. But these papers do not consider the large deflections of plates. To the authors' knowledge, no paper has been devoted to the investigations of non-linear behaviours of

† Formerly head of the Department of Mathematics, Government Engineering College, Jalpaiguri, West Bengal, India.

heated elastic skew plates having various applications in modern design, especially in the space industry.

In this paper non-linear behaviours of simply-supported heated skew plates (taken in rhombic form for simplicity of calculation) are investigated. Various numerical results have been calculated showing central deflection parameters versus thermal load functions. Whereas the results for skew angles other than 0° are believed to be new, the results for a 0°-skew angle are found to be in remarkable agreement with the already known results [see Biswas (1975)].

ANALYSIS

Let us consider a rhombic plate of skew angle θ whose uniform thickness is h and edge-length $2a$. The material of the plate is considered isotropic having mass density ρ, Young's modulus E and Poisson's ratio ν. The origin of the co-ordinates is located at the geometric centre of the plate. The deflections are consideed to be of the same order of magnitude as the plate thickness, the edge-length being sufficiently large compared to the thickness.

Now the uncoupled set of differential equations in rectangular Cartesian co-ordinates, governing the thermal behaviours of elastic plates [see Banerjee (1984)] is given by

$$\nabla^4 w - \frac{12A}{h^2}\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)-\frac{6\lambda}{h^2}\left[\nabla^2 w\left\{\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right\}\right.$$
$$\left.+2\left\{\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial w}{\partial x}\right)^2+\frac{\partial^2 w}{\partial y^2}\left(\frac{\partial w}{\partial y}\right)^2\right\}+4\frac{\partial^2 w}{\partial x\,\partial y}\cdot\frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y}\right]$$
$$+\frac{12\alpha_t\tau_0}{h^2}\sqrt{\lambda(1-\nu^2)}\cdot\nabla^2 w+(1+\nu)\alpha_t\nabla^2\tau=\frac{q}{D}, \quad (1)$$

where

$$A=\frac{\partial u}{\partial x}+\nu\frac{\partial v}{\partial y}+\frac{1}{2}\left\{\left(\frac{\partial w}{\partial x}\right)^2+\nu\left(\frac{\partial w}{\partial y}\right)^2\right\}-(1+\nu)\alpha_t\tau_0, \quad (2)$$

$\lambda=\nu^2$ for simply-supported elastic plates, and $D=Eh^3/12(1-\nu^2)$, the flexural rigidity of the material of the elastic plate.

It is to be noted that in the derivation of eqns (1) and (2) in rectangular Cartesian co-ordinates, the expression

$$(1-\nu^2)\left[\frac{\partial v}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]+\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y}\right)^2\cdot\frac{1}{2(1+\nu)}$$

in the total P.E. of the elastic plate (Banerjee, 1984) has been replaced by

$$\frac{\lambda}{4}\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]^2.$$

As a consequence the partial differential equations governing the deflection of the plate have become uncoupled and the two decoupled differential equations (1) and (2) have been obtained.

In the present problem, the temperature is assumed to vary linearly w.r.t. the thickness direction z. We also note that

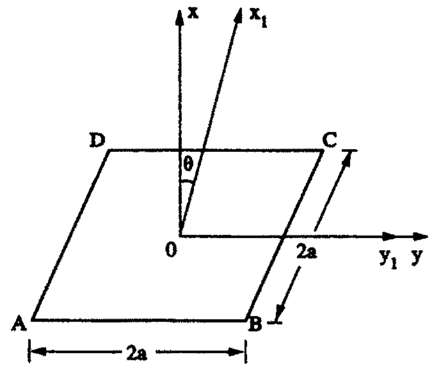

Fig. 1. Plan form of skew plate and co-ordinate system.

$$T(x,y,z) = \tau_0(x,y)+z\tau(x,y),$$

in which

$$\tau_0 = \tfrac{1}{2}(T_1+T_2), \quad \tau = \frac{1}{h}(T_1-T_2),$$

$$T_1 = T\left(x,y,\frac{h}{2}\right) \quad \text{and} \quad T_2 = T\left(x,y,-\frac{h}{2}\right) \quad \text{(Banerjee, 1984).}$$

Clearly τ_0 is the temperature in the middle plane and τ varies along the thickness of the plate and hence $\tau \neq \tau_0$.

The plan of the skew co-ordinates (x_1, y_1, θ) is shown in Fig. 1. Clearly

$$x = x_1 \cos\theta$$
$$\text{and} \quad y = y_1 + x_1 \sin\theta \tag{3}$$

are the co-ordinate transformation equations. Hence we have the following partial differential operators in oblique co-ordinates:

$$\frac{\partial}{\partial x} \equiv \sec\theta\left(\frac{\partial}{\partial x_1} - \sin\theta\frac{\partial}{\partial y_1}\right), \quad \frac{\partial}{\partial y} \equiv \frac{\partial}{\partial y_1},$$

$$\frac{\partial^2}{\partial x^2} \equiv \sec^2\theta\left(\frac{\partial^2}{\partial x_1^2} - 2\sin\theta\frac{\partial^2}{\partial x_1\,\partial y_1} + \sin^2\theta\frac{\partial^2}{\partial y_1^2}\right),$$

$$\frac{\partial^2}{\partial y^2} \equiv \frac{\partial^2}{\partial y_1^2}, \quad \frac{\partial^2}{\partial x\,\partial y} \equiv \sec\theta\left(\frac{\partial^2}{\partial x_1\,\partial y_1} - \sin\theta\frac{\partial^2}{\partial y_1^2}\right),$$

$$\nabla^2 \equiv \sec^2\theta\left(\frac{\partial^2}{\partial x_1^2} - 2\sin\theta\frac{\partial^2}{\partial x_1\,\partial y_1} + \frac{\partial^2}{\partial y_1^2}\right)$$

and

$$\nabla^4 \equiv \sec^4\theta\left\{\frac{\partial^4}{\partial x_1^4} - 4\sin\theta\left(\frac{\partial^4}{\partial x_1^3\,\partial y_1} + \frac{\partial^4}{\partial x_1\,\partial y_1^3}\right) + 2(1+2\sin^2\theta)\frac{\partial^4}{\partial x_1^2\,\partial y_1^2} + \frac{\partial^4}{\partial y_1^4}\right\}. \tag{4}$$

We now transform eqn (2) in oblique co-ordinates. For simply-supported plates the boundary conditions are

$$w = 0 \quad \text{at} \quad x_1 = \pm a \quad \text{and at} \quad y_1 = \pm a,$$

$$\frac{\partial^2 w}{\partial x_1^2} = 0 \quad \text{at} \quad x_1 = \pm a \quad \text{and} \quad \frac{\partial^2 w}{\partial y_1^2} = 0 \quad \text{at} \quad y_1 = \pm a.$$

Then let us choose the deflection function for the simply-supported plate as

$$w = w_0 \cos\frac{\pi x_1}{2a} \cos\frac{\pi y_1}{2a}, \tag{5}$$

which clearly satisfies the above-mentioned boundary conditions.

Now putting (5) in eqn (2) transformed in oblique co-ordinates and then integrating the relation thus obtained, over the entire surface of the plate, we obtain the value of A in the following form:

$$A = \frac{\pi^2 w_0^2}{32a^2}(1+\nu+2\tan^2\theta) - (1+\nu)\alpha_t\tau_0. \tag{6}$$

(As the normal displacement w is our primary interest, the in-plane displacements u, v have been eliminated through integration by the choice of appropriate functions for such displacements.) Again transforming eqn (1) in oblique co-ordinates, introducing eqns (5) and (6) in the transformed equation and then applying Galerkin's error minimizing technique we get the following equation determining the central deflection parameter w_0/h depending on the thermal load function $q'a^4/Eh^4$:

$$\left[(1+2\tan^2\theta)\sec^2\theta - \frac{6S}{(1+\nu)\pi^2}\{2\sqrt{\lambda(1-\nu^2)}\cdot\sec^2\theta + (1+\nu)(1+\nu+2\tan^2\theta)\}\right]\left(\frac{w_0}{h}\right) + \tfrac{3}{8}[(1+\nu+2\tan^2\theta)^2 + \frac{\lambda}{4}(8+49\tan^2\theta+29\tan^4\theta)]\left(\frac{w_0}{h}\right)^3 = \frac{768(1-\nu^2)}{\pi^6}\left(\frac{q'a^4}{Eh^4}\right), \tag{7}$$

where

$$S = 2\left(\frac{a}{h}\right)^2(1+\nu)\alpha_t\tau_0$$

and

$$q' = q - D\alpha_t(1+\nu)\nabla^2\tau.$$

Equation (7) is applicable for the immovable edge condition of the simply-supported skew plate. For the movable edge condition we have $A = 0$, so that eqn (7) takes the form:

$$\left[(1+2\tan^2\theta)\sec^2\theta - \frac{12S}{(1+\nu)\pi^2}\sqrt{\lambda(1-\nu^2)}\cdot\sec^2\theta\right]\left(\frac{w_0}{h}\right) + \frac{3\lambda}{32}(8+49\tan^2\theta+29\tan^4\theta)\left(\frac{w_0}{h}\right)^3 = \frac{768(1-\nu^2)}{\pi^6}\left(\frac{q'a^4}{Eh^4}\right). \tag{8}$$

NUMERICAL RESULTS

Numerical results are presented here (Tables 1 and 2) in the tabular forms for $S = 0$, 0.1 ; $\theta = 0°$, 15°, 30° and $q'a^4/Eh^4 = 2, 4, 8, 10$.

Table 1. $S = 0$, i.e. $\tau_0 = 0$

	w_0/h by present method						w_0/h by Berger's method†		
	$\theta = 0°$		$\theta = 15°$		$\theta = 30°$		(Biswas, 1975) $\theta = 0°$	$\theta = 15°$	$\theta = 30°$
$\frac{q'a^4}{Eh^4}$	Movable edge	Immovable edge	Movable edge	Immovable edge	Movable edge	Immovable edge	Immovable edge	Immovable edge	Immovable edge
2	1.30156	0.91435	1.08167	0.82069	0.6269	0.53604	0.9013	0.79972	0.53671
4	2.1909	1.3131	1.85443	1.20857	1.14734	0.84631	1.29017	1.16888	0.848
8	3.23354	1.78866	2.8581	1.67119	1.89675	1.22355	1.75406	1.60902	1.2266
10	3.73498	1.9613	3.2243	1.83866	2.17977	1.3597	1.92254	1.76847	1.36324

† Berger's method has been applied to the present problem by neglecting e_2, the second strain invariant in the expression for total P.E. of the plate.

Table 2. $S = 0.1$, i.e. $\tau_0 \neq 0$

	w_0/h by present method						w_0/h by Berger's method ($e_2 = 0$)		
	$\theta = 0°$		$\theta = 15°$		$\theta = 30°$		(Biswas, 1975) $\theta = 0°$	$\theta = 15°$	$\theta = 30°$
$\frac{q'a^4}{Eh^4}$	Movable edge	Immovable edge	Movable edge	Immovable edge	Movable edge	Immovable edge	Immovable edge	Immovable edge	Immovable edge
2	1.32786	0.94985	1.10168	0.83899	0.63597	0.55925	0.94058	0.83515	0.56109
4	2.22082	1.34324	1.87831	1.20992	1.1604	0.86901	1.32336	1.19954	0.87185
8	3.35106	1.81316	2.88067	1.65221	1.9111	1.24302	1.781	1.63412	1.24706
10	3.76118	1.98415	3.24585	1.81269	2.19385	1.37799	1.94764	1.79188	1.38247

OBSERVATIONS AND CONCLUSIONS

From the numerical analysis of the undertaken problem the following observations are made:

(i) The nature of the central deflection of a skew plate under thermal loading is the same as that of the plate under mechanical loading, i.e. the central deflection increases continuously with the increase of loading for any edge condition of the skew plate, whether movable or immovable.

(ii) The central deflection for the movable edge condition of the skew plate is always greater than that for the immovable edge condition of the plate, for the same loading.

(iii) Irrespective of the edge condition, the central deflection decreases with the increase in the skew angle.

(iv) The results for immovable edge conditions of the skew plate obtained by the present method, agree well with the results obtained by Berger's method. It is to be noted that Berger's method is a purely approximate method based on the neglect of e_2. But the present study is based on Banerjee's hypothesis which suggests a modified strain–energy expression, and thus this model embraces less approximation (Banerjee and Dutt, 1981) than that of Berger. Again Berger's method is meaningful only for immovable edge conditions of the plates.

(v) The deflections increase with τ_0.

The present method seems to be more advantageous than any other method found in open literature. The main advantages are:

(1) The differential equations are decoupled and easy to solve;
(2) from a single cubic equation determining w_0/h, the results could be obtained for movable as well as immovable edge conditions; and
(3) unlike Berger's method it gives accurate results both for movable and immovable edge conditions. Based on Banerjee's hypothesis a good number of works have been carried out and in each case sufficiently accurate results have been obtained [e.g. Banerjee and Dutt (1981), Banerjee (1984), Sinharay and Banerjee (1985) and Ray *et al.* (1992, 1993)]. So in the present case also, the same degree of accuracy was expected.

REFERENCES

Alec, B. J. (1949). Thermal stresses in a rectangular plate clamped along an edge. *J. Appl. Mech., Trans ASME* **71**.

Banerjee, B. (1984). Large deflections of polygonal plates under nonstationary temperature. *J. Thermal Stresses (U.S.A.)* **7**, 285–292.

Banerjee, B. and Dutt, S. (1979). Large deflections of elastic plates under non-stationary temperature. *J. ASCE* **4**, 705.

Banerjee, B. and Dutt, S. (1981). A new approach to an analysis of large deflections of thin elastic plates. *Int. J. Non-Lin. Mech.* **16**, 47–52.

Biswas, P. (1974). Large deflection of heated circular plate under non-stationary temperature. *Bull. Cal. Math. Soc.* **66**, 247–252.

Biswas, P. (1975). Large deflections of heated elastic plates under uniform load. *Mechanique Applique* **20**(4).

Boley, B. A. and Weiner, J. H. (1960). *Theory of Thermal Stresses* (2nd Edn). Wiley, New York.

Forray, M. and Newmann, M. (1960). Axisymmetric bending stresses in solid circular plates with thermal gradient. *J. Aerospace Sci.* **27**(9).

Kaiuk, la. F. and Pavlenko, V. I. (1971). Thermal buckling of parallelogram shaped plates (in Russian). *Teplovye Napriazheniia v Elementakh Konstruktsii* **11**, 173.

Kaiuk, la. F. and Pavlenko, V. I. (1972). Thermal stability of skewed plates (in Russian). *Voprosy Dinamiki i Prochnosti* **22**, 159.

Kamiya, N. (1978). Analysis of the large thermal bending of sandwich plates by a modified Berger method. *J. Strain Anal.* **13**(1), 17–22.

Katayama, T., Matumoto, E. and Sekiya, T. (1967). Fundamental equations for thermoelastic deformation of skew plates. *Bulletin of the University of OSAKA Prefecture, Series A* **16**.

Matumoto, E. and Sekiya, T. (1975). Elastic stability of thermally stressed parallelogram panels. *Trans. JSME* **41**(343), 736–745.

Nowacki, W. (1962). *Thermoelasticity*. Pergamon Press, Oxford.

Prabhu, M. S. S. and Durvasula, S. (1974). Elastic stability of thermally stressed clamped–clamped skew plates. *J. Appl. Mech., Trans ASME* **41**(3), 820–821.

Ray, A. K., Banerjee, B. and Bhattacharjee, B. (1992). Large deflections of rhombic plates—a new approach. *Int. J. Non-Lin. Mech.* **27**(6), 1007–1014.

Ray, A. K., Banerjee, B. and Bhattacharjee, B. (1993). *Meccanica* (in press).

Roychowdhury, S. K. (1972). Some problems on thermoelasticity. Ph.D. Thesis presented to the Jadavpur University, India.

Sarkar, S. R. (1968). Quasi-static thermal deflections in a solid circular plate in the axisymmetric case. *Aplikace Mathematiky, Czechoslovakia* **13**.

Schneider, P. J. (1955). Variation of maximum thermal stress in free plates. *J. Aerospace Sci.* **22**.

Sinharay, G. C. and Banerjee, B. (1985). A new approach to large deflection analysis of spherical and cylindrical shells under thermal loading. *Mech. Res. Comm. (U.S.A.)* **12**(2), 53–64.

Williams, M. L. (1955). Large deflection analysis for a plate strip subjected to normal pressure and heat. *J. Appl. Mech., Trans ASME* **22**(4).

Williams, M. L. (1958). Further large deflection analysis for a plate strip subjected to normal pressure and heating. *J. Appl. Mech., Trans ASME* **25**(2).

Zizicas, G. A. (1952). Transient thermal stresses in thin isotropic plates. *ICLA Engng Report* **52**(7).